

Pairings of Sheaves of \mathcal{A} -Modules through Bilinear \mathcal{A} -Morphisms

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Abstract

It is proved that for any free \mathcal{A} -modules \mathcal{F} and \mathcal{E} of finite rank on some \mathbb{C} -algebraized space (X, \mathcal{A}) a *degenerate* bilinear \mathcal{A} -morphism $\Phi : \mathcal{F} \times \mathcal{E} \longrightarrow \mathcal{A}$ induces a *non-degenerate* bilinear \mathcal{A} -morphism $\overline{\Phi} : \mathcal{F}/\mathcal{E}^\perp \times \mathcal{E}/\mathcal{F}^\perp \longrightarrow \mathcal{A}$, where \mathcal{E}^\perp and \mathcal{F}^\perp are the *orthogonal* sub- \mathcal{A} -modules associated with \mathcal{E} and \mathcal{F} , respectively. This result generalizes the finite case of the classical result, which states that given two vector spaces W and V , paired into a field k , the induced vector spaces W/V^\perp and V/W^\perp have the same dimension. Some related results are discussed as well.

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Introduction

The goal of this paper is to provide additional steps for developing *classical symplectic geometry* within the setting of *Abstract Differential Geometry*

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(*ADG* in short) (*à la* Mallios), cf. [3] and [4]. The attempt of taking *ADG* to new horizons, such those related to classical symplectic geometry was initiated in the our paper [5]. The main result in the article [5] is that given an ordered \mathbb{R} -algebraized space $(X, \mathcal{A}, \mathcal{P}, |\cdot|)$, the standard free \mathcal{A} -module \mathcal{A}^n of rank n on X , there exists for every open subset U of X a basis $\mathcal{B}(U)$ of $\mathcal{A}^n(U)$, relative to which the matrix of a non-zero skew-symmetric and non-degenerate bilinear sheaf \mathcal{A} -morphism $\omega \equiv (\omega_U) : \mathcal{A}^n \oplus \mathcal{A}^n \longrightarrow \mathcal{A}$ is the matrix

$$\begin{pmatrix} 0 & \mathbf{I}_n \\ -\mathbf{I}_n & 0 \end{pmatrix}.$$

In order to aptly pursue the goal of our ongoing research project, we are laying some ground work, regarding pairings of sheaves of \mathcal{A} -modules. Duality and pairings of \mathcal{A} -modules, as we will see in our subsequent work, are a necessary prerequisite for the layout of Abstract Symplectic Geometry. In this article, the most important results are contained in Theorems 2.3 and 2.4. Theorem 2.3 contends that given \mathcal{A} -modules \mathcal{F} and \mathcal{E} , paired into \mathcal{A} such that the left kernel $\mathcal{E}^\perp = 0$, one may find natural \mathcal{A} -isomorphisms into

$$\mathcal{E}/\mathcal{F}_0^\perp \longrightarrow \mathcal{F}_0^*, \quad \text{and} \quad \mathcal{E}_0^\perp \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*$$

for sub- \mathcal{A} -modules \mathcal{F}_0 and \mathcal{E}_0 of \mathcal{F} and \mathcal{E} , respectively. In the special case where \mathcal{F} and \mathcal{E} are free \mathcal{A} -modules of finite rank, if $\Phi : \mathcal{F} \times \mathcal{E} \longrightarrow \mathcal{A}$ is bilinear, then \mathcal{F} is \mathcal{A} -isomorphic to \mathcal{E} provided that Φ is non-degenerate, and if Φ is degenerate, it induces a non-degenerate \mathcal{A} -morphism

$$\overline{\Phi} : \mathcal{F}/\mathcal{E}^\perp \times \mathcal{E}/\mathcal{F}^\perp \longrightarrow \mathcal{A}$$

such that

$$(\mathcal{F}/\mathcal{E}^\perp)^\perp = 0 = (\mathcal{E}/\mathcal{F}^\perp)^\perp.$$

1 Theorems on ranks of free \mathcal{A} -modules

Let $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ and $\mathcal{F} \equiv (\mathcal{F}, \rho, X)$ be two \mathcal{A} -modules on a topological space X , and let

$$\Gamma(\mathcal{E}) \equiv (\Gamma(\mathcal{E})(U) \equiv \Gamma(U, \mathcal{E}) \equiv \mathcal{E}(U), \pi_V^U)$$

and

$$\Gamma(\mathcal{F}) \equiv (\mathcal{F}(U), \rho_V^U)$$

be the complete presheaves of sections of \mathcal{E} and \mathcal{F} , respectively. For every open set $U \subseteq X$,

$$(\mathcal{E} + \mathcal{F})(U) \equiv \mathcal{E}(U) + \mathcal{F}(U) := \Gamma(U, \mathcal{E}) + \Gamma(U, \mathcal{F})$$

is the sum of $\mathcal{A}(U)$ -modules $\mathcal{E}(U)$ and $\mathcal{F}(U)$, where $\Gamma(\mathcal{A}) \equiv (\Gamma(U, \mathcal{A}), \kappa_V^U)$ is the complete presheaf of sections of the coefficient sheaf \mathcal{A} . Next, consider the following presheaf on X , defined by the rule

$$U \longmapsto S(U) := \mathcal{E}(U) + \mathcal{F}(U), \quad (1)$$

where U is open in X and restriction maps are maps $\lambda_V^U : S(U) \longrightarrow S(V)$ such that

$$\lambda_V^U(s + t) = \pi_V^U(s) + \rho_V^U(t),$$

for all $s \in \mathcal{E}(U)$ and $t \in \mathcal{F}(U)$. Thus, (1) yields a $\Gamma(\mathcal{A})$ -presheaf on X , which we denote

$$S \equiv \Gamma(\mathcal{E} + \mathcal{F}) := \Gamma(\mathcal{E}) + \Gamma(\mathcal{F}).$$

The \mathcal{A} -module

$$\mathcal{E} + \mathcal{F} := \mathbf{S}(S) \equiv \mathbf{S}(\Gamma(\mathcal{E}) + \Gamma(\mathcal{F}))$$

generated by the presheaf S is called the *sum* of \mathcal{A} -modules \mathcal{E} and \mathcal{F} .

The reader will have no difficulty in proving that the sheaf $\mathcal{E} + \mathcal{F}$ is indeed an \mathcal{A} -module, and S is complete.

Theorem 1.1 *Let \mathcal{E} be an \mathcal{A} -module on a topological space X , \mathcal{F} and \mathcal{G} be sub- \mathcal{A} -modules of \mathcal{E} . Then,*

$$(\mathcal{F} + \mathcal{G})/\mathcal{F} = \mathcal{G}/(\mathcal{F} \cap \mathcal{G})$$

within an \mathcal{A} -isomorphism.

Proof. Let $\phi \equiv (\phi_U)_{U \in \tau}$ be the (canonical) quotient $\Gamma(\mathcal{A})$ -morphism

$$\Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E})/\Gamma(\mathcal{F}),$$

where $\Gamma(\mathcal{E}) \equiv (\mathcal{E}(U), \pi_V^U)$ is the complete presheaf of sections of \mathcal{E} , and

$$\Gamma(\mathcal{E})/\Gamma(\mathcal{F}) \equiv ((\Gamma(\mathcal{E})/\Gamma(\mathcal{F}))(U) \equiv \Gamma(U, \mathcal{E})/\Gamma(U, \mathcal{F}) \equiv \mathcal{E}(U)/\mathcal{F}(U), \sigma_V^U)$$

(the σ_V^U are the obvious restriction maps, given by

$$\sigma_V^U(s + \Gamma(U, \mathcal{F})) = \pi_V^U(s) + \Gamma(V, \mathcal{F}),$$

for all $s \in \Gamma(U, \mathcal{E})$) is the generating presheaf of the quotient \mathcal{A} -module \mathcal{E}/\mathcal{F} , cf. Mallios[[3], pp. 114, 115]. For every open $U \subseteq X$, the restriction ψ_U of ϕ_U to the sub- $\mathcal{A}(U)$ -module $\Gamma(U, \mathcal{G})$ is the canonically constructed $\mathcal{A}(U)$ -morphism

$$\psi_U : \mathcal{G}(U) \longrightarrow \mathcal{E}(U)/\mathcal{F}(U),$$

given by

$$\psi_U(s) := s + \mathcal{F}(U), \quad s \in \mathcal{G}(U).$$

For every fixed open $U \subseteq X$, the union of the cosets $s + \mathcal{F}(U)$, $s \in \mathcal{G}(U)$, forms the $\mathcal{A}(U)$ -module $\mathcal{G}(U) + \mathcal{F}(U) \equiv (\mathcal{G} + \mathcal{F})(U)$; therefore

$$\psi_U(\mathcal{G}(U)) = (\mathcal{G}(U) + \mathcal{F}(U))/\mathcal{F}(U) \equiv (\mathcal{G} + \mathcal{F})(U)/\mathcal{F}(U).$$

But, for all elements $s \in \mathcal{G}(U)$, with U as above, i.e. an open set in X , we have

$$\psi_U(s) = \phi_U(s)$$

and

$$\ker \phi_U = \mathcal{F}(U).$$

Therefore,

$$\ker \psi_U = \mathcal{F}(U) \cap \mathcal{G}(U) \equiv (\mathcal{F} \cap \mathcal{G})(U).$$

By elementary algebra, we construct the canonical $\mathcal{A}(U)$ -isomorphism

$$\bar{\psi}_U : \mathcal{G}(U)/(\mathcal{F} \cap \mathcal{G})(U) \longrightarrow (\mathcal{F} + \mathcal{G})(U)/\mathcal{F}(U),$$

for every open $U \subseteq X$. If $V \subseteq U$ is an open subset of a certain fixed open $U \subseteq X$, it is not difficult to see that the diagram

$$\begin{array}{ccc} \mathcal{G}(U)/(\mathcal{F} \cap \mathcal{G})(U) & \xrightarrow{\bar{\psi}_U} & (\mathcal{F} + \mathcal{G})(U)/\mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{G}(V)/(\mathcal{F} \cap \mathcal{G})(V) & \xrightarrow{\bar{\psi}_V} & (\mathcal{F} + \mathcal{G})(V)/\mathcal{F}(V) \end{array}$$

commutes. Consequently, $\bar{\psi} \equiv (\bar{\psi}_U)_{U \in \tau}$ is a $\Gamma(\mathcal{A})$ -isomorphism of the presheaves $\Gamma(\mathcal{G})/\Gamma(\mathcal{F} \cap \mathcal{G})$ and $\Gamma(\mathcal{F} + \mathcal{G})/\Gamma(\mathcal{F})$. Applying the sheafification functor \mathbf{S} (see Mallios[[3], p. 33]) to the diagram

$$\bar{\psi} \equiv (\bar{\psi}_U) : \Gamma(\mathcal{G})/\Gamma(\mathcal{F} \cap \mathcal{G}) \longrightarrow \Gamma(\mathcal{F} + \mathcal{G})/\Gamma(\mathcal{F}),$$

we obtain the sought \mathcal{A} -isomorphism

$$\bar{\psi} : \mathcal{G}/(\mathcal{F} \cap \mathcal{G}) \longrightarrow (\mathcal{F} + \mathcal{G})/\mathcal{F}.$$

■

In the special case that $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$, it follows for any open $U \subseteq X$, cf. Mallios[[3], pp. 121, 122], that

$$\Gamma(U, \mathcal{E}) = \Gamma(U, \mathcal{F}) \oplus \Gamma(U, \mathcal{G}).$$

Applying Artin[[1], Theorem 1.2, p. 7], we obtain that $\Gamma(U, \mathcal{E})/\Gamma(U, \mathcal{F})$ is $\mathcal{A}(U)$ -isomorphic to $\Gamma(U, \mathcal{G})/0 = \Gamma(U, \mathcal{G})$, where U is any arbitrary open set in X . It thus follows that

$$\mathcal{E}/\mathcal{F} := \mathbf{S}(\Gamma(\mathcal{E})/\Gamma(\mathcal{F})) = \mathbf{S}(\Gamma(\mathcal{G})) = \mathcal{G},$$

that is

$$\mathcal{E}/\mathcal{F} = \mathcal{G}$$

within an \mathcal{A} -isomorphism.

Before we proceed to some more theorems on \mathcal{A} -modules, let us define what is meant by *free sub- \mathcal{A} -modules of a free \mathcal{A} -module*.

Definition 1.1 Let \mathcal{E} be the free \mathcal{A} -module $\mathcal{A}^{(I)} := \oplus_I \mathcal{A}$, where I is an arbitrary indexing set, and let $\mathcal{F} \subseteq \mathcal{E}$ be a sub- \mathcal{A} -module of \mathcal{E} such that

$$\mathcal{F}_x := \mathcal{A}_x^{(J)} \oplus \underbrace{0 \oplus \dots \oplus 0}_{I \setminus J} \subseteq \mathcal{A}_x^{(I)} =: \mathcal{E}_x$$

for all $x \in X$, and where J is a subset of I . \mathcal{F} is called a *free sub- \mathcal{A} -module* of \mathcal{E} , and is easily identified with the free \mathcal{A} -module $\mathcal{A}^{(J)}$. The free sub- \mathcal{A} -module $\mathcal{G} := \mathcal{A}^{(I \setminus J)}$ is called a free sub- \mathcal{A} -module *supplementary to \mathcal{F}* . It is obvious that $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$. The fibers of \mathcal{G} are \mathcal{A}_x -modules

$$\mathcal{G}_x = \underbrace{0 \oplus \dots \oplus 0}_I \oplus \mathcal{A}_x^{(I \setminus J)} \subseteq \mathcal{A}_x^I.$$

□

Theorem 1.4, cf. Artin[[1], p. 9], is also immediate. In effect, let \mathcal{E} be a free \mathcal{A} -module on a topological space X , viz. $\mathcal{E} \cong \mathcal{A}^{(I)}$, where I is arbitrary, and let \mathcal{F} be a free sub- \mathcal{A} -module of \mathcal{E} . If \mathcal{G} is a free sub- \mathcal{A} -module supplementary to \mathcal{F} in \mathcal{E} , we have that

$$\text{rank } \mathcal{E} = \text{rank } \mathcal{F} + \text{rank } \mathcal{G},$$

and since $\mathcal{G} \cong \mathcal{E}/\mathcal{F}$, one obtains

$$\text{rank } \mathcal{E} = \text{rank } \mathcal{F} + \text{rank } \mathcal{E}/\mathcal{F}.$$

Let now $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$ be free sub- \mathcal{A} -modules of a free \mathcal{A} -module \mathcal{E} . For some free sub- \mathcal{A} -modules \mathcal{G}_2 and \mathcal{G}_3 of \mathcal{E} , we have

$$\mathcal{F}_2 = \mathcal{F}_1 \oplus \mathcal{G}_2, \quad \mathcal{F}_3 = \mathcal{F}_2 \oplus \mathcal{G}_3$$

and, therefore,

$$\mathcal{F}_3 = \mathcal{F}_1 \oplus (\mathcal{G}_2 \oplus \mathcal{G}_3).$$

By virtue of Theorem 1.1, \mathcal{G}_2 , \mathcal{G}_3 and $\mathcal{G}_2 \oplus \mathcal{G}_3$ are \mathcal{A} -isomorphic to $\mathcal{F}_2/\mathcal{F}_1$, $\mathcal{F}_3/\mathcal{F}_2$ and $\mathcal{F}_3/\mathcal{F}_1$, respectively; consequently

$$\begin{aligned} \text{rank } \mathcal{G}_2 &= \text{rank } \mathcal{F}_2/\mathcal{F}_1 \\ \text{rank } \mathcal{G}_3 &= \text{rank } \mathcal{F}_3/\mathcal{F}_2 \\ \text{rank } (\mathcal{G}_2 \oplus \mathcal{G}_3) &= \text{rank } \mathcal{F}_3/\mathcal{F}_1. \end{aligned}$$

Thus, we obtain

$$\text{rank } \mathcal{F}_3/\mathcal{F}_1 = \text{rank } \mathcal{F}_2/\mathcal{F}_1 + \text{rank } \mathcal{F}_3/\mathcal{F}_2. \quad (2)$$

Let now \mathcal{F} and \mathcal{G} be two given free sub- \mathcal{A} -modules of a free \mathcal{A} -module \mathcal{E} . For $\mathcal{F}_1 = 0$, $\mathcal{F}_2 = \mathcal{F}$, $\mathcal{F}_3 = \mathcal{F} + \mathcal{G}$, Equation (2) becomes

$$\begin{aligned} \text{rank } (\mathcal{F} + \mathcal{G}) &= \text{rank } \mathcal{F} + \text{rank } (\mathcal{F} + \mathcal{G})/\mathcal{F} \\ &= \text{rank } \mathcal{F} + \text{rank } \mathcal{G}/(\mathcal{F} \cap \mathcal{G}). \end{aligned}$$

Adding $\text{rank } (\mathcal{F} \cap \mathcal{G})$ to both sides of the last equation, and using the fact that

$$\text{rank } \mathcal{G} = \text{rank } (\mathcal{F} \cap \mathcal{G}) + \text{rank } \mathcal{G}/(\mathcal{F} \cap \mathcal{G}),$$

we obtain

$$\text{rank } (\mathcal{F} + \mathcal{G}) + \text{rank } (\mathcal{F} \cap \mathcal{G}) = \text{rank } \mathcal{F} + \text{rank } \mathcal{G}.$$

Next, we put in Equation (2) $\mathcal{F}_1 = \mathcal{F} \cap \mathcal{G}$, $\mathcal{F}_2 = \mathcal{G}$, $\mathcal{F}_3 = \mathcal{E}$ to get

$$\begin{aligned} \text{rank } \mathcal{E}/(\mathcal{F} \cap \mathcal{G}) &= \text{rank } \mathcal{G}/(\mathcal{F} \cap \mathcal{G}) + \text{rank } \mathcal{E}/\mathcal{G} \\ &= \text{rank } (\mathcal{F} + \mathcal{G})/\mathcal{F} + \text{rank } \mathcal{E}/\mathcal{G}. \end{aligned}$$

If we add $\text{rank } \mathcal{E}/(\mathcal{F} + \mathcal{G})$, and use the equation

$$\text{rank } \mathcal{E}/\mathcal{F} = \text{rank } (\mathcal{F} + \mathcal{G})/\mathcal{F} + \text{rank } \mathcal{E}/(\mathcal{F} + \mathcal{G})$$

(Put $\mathcal{F}_1 = \mathcal{F}$, $\mathcal{F}_2 = \mathcal{F} + \mathcal{G}$, $\mathcal{F}_3 = \mathcal{E}$ in Equation (2) to obtain the last equation.), we obtain

$$\text{rank } \mathcal{E}/(\mathcal{F} + \mathcal{G}) + \text{rank } \mathcal{E}/(\mathcal{F} \cap \mathcal{G}) = \text{rank } \mathcal{E}/\mathcal{F} + \text{rank } \mathcal{E}/\mathcal{G}.$$

Definition 1.2 Let \mathcal{E} be a free \mathcal{A} -module on a topological space X , and \mathcal{F} a free sub- \mathcal{A} -module of \mathcal{E} such that its supplement is a free sub- \mathcal{A} -module of *finite rank*. The rank of the free sub- \mathcal{A} -module \mathcal{E}/\mathcal{F} is called the *corank* of \mathcal{F} , viz.

$$\text{corank } \mathcal{F} = \text{rank } \mathcal{E}/\mathcal{F}.$$

□

The above various results can be expressed as follows:

Theorem 1.2 Let \mathcal{E} be a free \mathcal{A} -module on a topological space X , and \mathcal{F} and \mathcal{G} free sub- \mathcal{A} -modules of \mathcal{E} . Then,

$$\begin{aligned} \text{rank } \mathcal{F} + \text{corank } \mathcal{F} &= \text{rank } \mathcal{E} \\ \text{rank } (\mathcal{F} + \mathcal{G}) + \text{rank } (\mathcal{F} \cap \mathcal{G}) &= \text{rank } \mathcal{F} + \text{rank } \mathcal{G} \\ \text{corank } (\mathcal{F} + \mathcal{G}) + \text{corank } (\mathcal{F} \cap \mathcal{G}) &= \text{corank } \mathcal{F} + \text{corank } \mathcal{G}. \end{aligned}$$

2 Pairings

Definition 2.1 Let (X, \mathcal{A}) be a \mathbb{C} -algebraized space, and let \mathcal{E} and \mathcal{F} be \mathcal{A} -modules on X . We say that \mathcal{F} and \mathcal{E} are **paired into** \mathcal{A} , or $[\mathcal{F}, \mathcal{E}; \mathcal{A}]$ is a **pairing** provided a *bilinear* $\Gamma(\mathcal{A})$ -morphism $\vartheta : \Gamma(\mathcal{F}) \oplus \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{A})$ is defined between the $\Gamma(\mathcal{A})$ -presheaves $\Gamma(\mathcal{F}) \oplus \Gamma(\mathcal{E})$ and $\Gamma(\mathcal{A})$. \square

We notice for any \mathcal{A} -module \mathcal{E} on X , \mathcal{E}^* and \mathcal{E} are paired into \mathcal{A} by the bilinear $\Gamma\mathcal{A}$ -morphism $\vartheta : \Gamma(\mathcal{E}^*) \oplus \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{A})$, given by

$$\vartheta_U(\phi, s) := \phi(s)$$

for $\phi \equiv (\phi_V)_{U \supseteq V, \text{ open}} \in \mathcal{E}^*(U) := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})(U) \equiv Hom_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)$ (cf. Mallios[[3], relation (5.1), p. 298, Definition 6.1, p.134]), $s \in \mathcal{E}(U)$, and U an open subset of X .

Now, consider a free \mathcal{A} -module \mathcal{E} , i.e. $\mathcal{E} = \mathcal{A}^{(I)}$ within an \mathcal{A} -isomorphic, and let $\{s_i\}$ be a basis of the $\mathcal{A}(U)$ -module $\mathcal{E}(U)$, where U is an open subset of X . (Cf. Mallios[[3], (3.13), p.122] For any open $U \subseteq X$, $\mathcal{E}(U)$ is $\mathcal{A}(U)$ -isomorphic to $\prod_{i \in I} \mathcal{A}(U) \equiv \prod_{i \in I} \Gamma(U, \mathcal{A})$.) Let $\phi \equiv (\phi_V)_{U \supseteq V, \text{ open}}$ be an element of $\mathcal{E}^*(U)$; put

$$\vartheta(\phi, s_i) \equiv \phi_U(s_i) := a_i \in \mathcal{A}(U),$$

for all $i \in I$. An element $s \in \mathcal{E}(U)$ is written uniquely in the form $s = \sum_{i \in I} s_i r_i$, where $0 \neq r_i \in \mathcal{A}(U)$ holds only for finitely many indices i . Then,

$$\vartheta(\phi, s) \equiv \phi_U(s) = \sum_{i \in I} \phi_U(s_i) r_i =: \sum_{i \in I} a_i r_i.$$

Now, assume that the restriction maps for the presheaf of sections of \mathcal{E} (resp. \mathcal{A}) are given by ρ_V^U (resp. σ_V^U), where V and U are open subsets of X , with $V \subseteq U$. For all $s \in \mathcal{E}(U)$, we have for any open $V \subseteq U$,

$$\phi_V(\rho_V^U(s)) \equiv \phi_V(s|_V) = \sigma_V^U(\phi_U(s)) = \sigma_V^U\left(\sum_{i \in I} a_i r_i\right) =: \sum_{i \in I} a_i|_V r_i|_V;$$

moreover, since $\{\rho_V^U(s_i)\}_{i \in I}$ is a basis of $\mathcal{E}(V)$, it follows that $\phi \equiv (\phi_V) \in \mathcal{E}(U)$ is known if all the $a_i \in \mathcal{A}(U)$ are known.

Select conversely an $a_i \in \mathcal{A}(U)$ for each index i , and define

$$\vartheta_U(\phi, s) := \phi_U(s) = \sum_{i \in I} a_i r_i,$$

assuming as above that $\phi \equiv (\phi_V)_{U \supseteq V, \text{ open}} \in \mathcal{E}^*(U)$, $s \in \mathcal{E}(U)$ and $s = \sum_{i \in I} s_i r_i$, where $\{s_i\}_{i \in I}$ is a basis of $\mathcal{E}(U)$. The sum $\sum_{i \in I} a_i r_i$ is finite since only finitely many r_i are non-zero. That $\phi_U(s + t) = \phi_U(s) + \phi_U(t)$, and $\phi_U(sa) = \phi_U(s)a$ for all $s, t \in \mathcal{E}(U)$, and $a \in \mathcal{A}(U)$ is immediately clear. Since $s_i = \sum_{j \in I} s_j \delta_{ji}$, where, as usual $\delta_{ii} = 1 \in \mathcal{A}(U)$ and $\delta_{ji} = 0$ for $j \neq i$, we obtain $\vartheta_U(\phi, s_i) := \phi_U(s_i) = \sum_{j \in I} a_j \delta_{ji} = a_i$. Thus,

Theorem 2.1 *Let (X, \mathcal{A}) be a \mathbb{C} -algebraized space and \mathcal{E} be a free \mathcal{A} -module. If $\{s_i\}$ is a basis of $\mathcal{E}(U)$, where U is open in X , then for arbitrarily chosen sections $a_i \in \mathcal{A}(U)$, there is one and only one $\phi \in \mathcal{E}^*(U)$ such that*

$$\phi_U(s_i) = a_i.$$

As above let \mathcal{E} be a free \mathcal{A} -module on a topological space X , viz. $\mathcal{E} = \mathcal{A}^{(I)}$ within an \mathcal{A} -isomorphism, and let $\{s_i\}$ be a basis of $\mathcal{E}(U) = \mathcal{A}^{(I)}(U) := \Gamma(U, \mathcal{A}^{(I)}) = \Gamma(U, \mathcal{A})^{(I)}$. Denote by ϕ_i the $\mathcal{A}|_U$ -morphism in $\mathcal{E}^*(U) := \text{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)$ for which $\phi_i(s_j) := \phi_{i,U}(s_j) = \delta_{ij,U}$. Let $B(U)$ be the sub- $\mathcal{A}|_U$ -module of $\mathcal{E}^*(U)$, spanned by the $\mathcal{A}|_U$ -morphisms ϕ_i . That is, given $\phi \in \mathcal{E}^*(U)$, $\phi \in B(U)$ provided $\phi = \sum_{i \in I} \alpha_i \phi_i$, where $\alpha_i \in \mathcal{A}|_U$ for all $i \in I$, and only finitely many α_i are non-zero. Since there are as many ϕ_i as there are s_i , we get that $\dim B(U) = \dim \mathcal{E}(U)$. If I is finite, $\mathcal{E}^* = \mathcal{E}$ within an \mathcal{A} -isomorphism (cf. Mallios[[3], p. 298]); therefore $\dim \mathcal{E}^*(U) = \dim \mathcal{E}(U)$ for any open subset $U \subseteq X$. If $\dim \mathcal{E}(U) = \infty$, then $\dim B(U) = \infty$ and, since $B(U) \subseteq \mathcal{E}^*(U)$, we may put $\dim \mathcal{E}^*(U) = \infty$. So in this case as well, we have

$$\dim \mathcal{E}(U) = \dim \mathcal{E}^*(U),$$

for any open $U \subseteq X$. Next, if $s = \sum_{i \in I} s_i a_i \in \mathcal{E}(U)$, where U is a fixed open subset of X , and $s \neq 0$, then at least one $a_j \neq 0$, so that $\phi_j(s) \neq 0$. By definition of \mathcal{E}^* , we know trivially that only the zero section vanishes on

all of $\mathcal{E}(U)$. Now, we see an analogue: If $s \in \mathcal{E}(U)$, and $\phi(s) = 0$ for all $\phi \in \mathcal{E}^*(U)$, then $s = 0$.

Hence, we have

Theorem 2.2 *Let \mathcal{E} be a free \mathcal{A} -module on a topological space X . Then, for any open subset $U \subseteq X$, $\dim \mathcal{E}^*(U) = \dim \mathcal{E}(U)$. If $\phi(s) = 0$ for all $s \in \mathcal{E}(U)$, then $\phi = 0$; on the other side, if $\phi(s) = 0$ for all $\phi \in \mathcal{E}^*(U)$, then $s = 0$. Finally, let $\dim \mathcal{E}(U) = n$ for some fixed open $U \subseteq X$, then $\dim \mathcal{E}(V) = n$ for any open $V \subseteq X$. To a given basis $\{s_i\}$ of $\mathcal{E}(U)$, we can find a dual basis $\{\phi_i\}$ of $\mathcal{E}^*(U) = \mathcal{E}(U)$, where*

$$\phi_i(s_j) := \phi_{i,U}(s_j) = \delta_{ij,U} \in \mathcal{A}(U).$$

Turning over to pairings of \mathcal{A} -modules, we suppose that \mathcal{A} -modules \mathcal{F} and \mathcal{E} , defined on a topological space X , are given and form a pairing into \mathcal{A} .

Definition 2.2 Let U be an open subset of X , $t \in \mathcal{F}(U)$ and $s \in \mathcal{E}(U)$. We say that t is **orthogonal** to s provided if $\vartheta \equiv (\vartheta_U)_{X \supseteq U, \text{ open}} : \Gamma(\mathcal{F}) \oplus \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{A})$ the bilinear $\Gamma(\mathcal{A})$ -morphism defining the pairing $[\mathcal{F}, \mathcal{E} : \mathcal{A}]$, then $\vartheta_U(t, s) := ts = 0$, i.e. $t(x)s(x) = 0$, for all $x \in U$. More generally, $\mathcal{F}(U)$ is said to be *orthogonal* to $\mathcal{E}(U)$ if $ts = 0$ for all $t \in \mathcal{F}(U)$ and $s \in \mathcal{E}(U)$. Similarly, a sub- \mathcal{A} -module \mathcal{F}_0 of \mathcal{F} is *orthogonal* to a sub- \mathcal{A} -module \mathcal{E}_0 of \mathcal{E} if $\mathcal{F}_0(U)$ is orthogonal to $\mathcal{E}_0(U)$ for any open set $U \subseteq X$. \square

Lemma 2.1 *Let $[\mathcal{F}, \mathcal{E}; \mathcal{A}]$ be a pairing in which \mathcal{F} and \mathcal{E} are \mathcal{A} -modules on a topological space X , and let \mathcal{F}_0 and \mathcal{E}_0 be sub- \mathcal{A} -modules of \mathcal{F} and \mathcal{E} , respectively. Furthermore, let*

$$\mathcal{F}_0(U)^\perp = \{s \in \mathcal{E}(U) : ts = 0 \text{ for all } t \in \mathcal{F}_0(U) \subseteq \mathcal{F}(U)\}$$

and

$$\mathcal{E}_0(U)^\perp = \{t \in \mathcal{F}(U) : ts = 0 \text{ for all } s \in \mathcal{E}_0(U) \subseteq \mathcal{E}(U)\}$$

for all open $U \subseteq X$, and let $(\mathcal{F}(U), \rho_V^U)$ and $(\mathcal{E}(U), \pi_V^U)$ be (complete) presheaves of sections of \mathcal{F} and \mathcal{E} , respectively. The **sheaf** generated by the presheaf, given by the correspondence

$$U \longmapsto \mathcal{E}_0(U)^\perp \quad (\text{resp. } U \longmapsto \mathcal{F}_0(U)^\perp),$$

where U is an open subset of X , along with restriction maps

$$(\pi_0^\perp)_V^U : \mathcal{E}_0(U)^\perp \longrightarrow \mathcal{E}_0(V)^\perp \quad (\text{resp. } (\rho_0^\perp)_V^U : \mathcal{F}_0(U)^\perp \longrightarrow \mathcal{F}_0(V)^\perp)$$

such that

$$(\pi_0^\perp)_V^U := \rho_V^U|_{\mathcal{E}_0(U)^\perp} \quad (\text{resp. } (\rho_0^\perp)_V^U := \pi_V^U|_{\mathcal{F}_0(U)^\perp})$$

is a **sub- \mathcal{A} -module** of \mathcal{F} (resp. \mathcal{E}), and is called the **sub- \mathcal{A} -module orthogonal** to \mathcal{E}_0 (resp. \mathcal{F}_0). We will denote by

$$\mathcal{E}_0^\perp \quad (\text{resp. } \mathcal{F}_0^\perp)$$

the **sub- \mathcal{A} -module orthogonal** to \mathcal{E}_0 (resp. \mathcal{F}_0), thus obtained.

Proof. For any open $U \subseteq X$, one sees easily that $\mathcal{E}_0(U)^\perp$ and $\mathcal{F}_0(U)^\perp$ are sub- $\mathcal{A}(U)$ -modules of $\mathcal{A}(U)$ -modules $\mathcal{F}(U)$ and $\mathcal{E}(U)$, respectively. It follows from Mallios-Ntumba[[6], Definition 1.1] that \mathcal{E}_0^\perp and \mathcal{F}_0^\perp are subsheaves of \mathcal{F} and \mathcal{E} , respectively. Finally, the sheafifications \mathcal{E}_0^\perp and \mathcal{F}_0^\perp , by virtue of Mallios[[3], Statement (1.54), p. 101], are sub- \mathcal{A} -modules of \mathcal{F} and \mathcal{E} , respectively. ■

Lemma 1.2 in Mallios-Ntumba[6] is a particular case of Lemma 2.2 below; the proof of Lemma 1.2 [6] applies here as well.

Lemma 2.2 *Let \mathcal{E} and \mathcal{F} be \mathcal{A} -modules on a topological space X , and suppose that $[\mathcal{F}, \mathcal{E}; \mathcal{A}]$ is a pairing. Then, for any sub- \mathcal{A} -module \mathcal{E}_0 of \mathcal{E} , the correspondence*

$$U \longmapsto \mathcal{E}_0(U)^\perp$$

along with maps $(\pi_0^\perp)_V^U$, as defined above, yields a complete presheaf of \mathcal{A} -modules on X . Similarly, for any sub- \mathcal{A} -module \mathcal{F}_0 of \mathcal{F} , the assignment

$$U \longmapsto \mathcal{F}_0(U)^\perp$$

with the afore-defined maps $(\rho_0^\perp)_V^U$ define a complete presheaf of \mathcal{A} -modules on X .

By virtue of Proposition 11.1, see Mallios[[3], p.51], if \mathcal{E}_0 and \mathcal{F}_0 are sub- \mathcal{A} -modules of \mathcal{E} and \mathcal{F} , respectively, where \mathcal{E} and \mathcal{F} form a pairing $[\mathcal{F}, \mathcal{E}; \mathcal{A}]$, then for any open $U \subseteq X$,

$$\mathcal{E}_0^\perp(U) = \mathcal{E}_0(U)^\perp \quad \text{and} \quad \mathcal{F}_0^\perp(U) = \mathcal{F}_0(U)^\perp$$

up to $\mathcal{A}(U)$ -isomorphisms. It is however trivial that $\mathcal{E}_0 \subseteq (\mathcal{E}_0^\perp)^\perp := \mathcal{E}_0^{\perp\perp}$. Similarly, $\mathcal{F}_0 \subseteq \mathcal{F}_0^{\perp\perp}$. Of special importance is the sub- \mathcal{A} -module \mathcal{E}^\perp of \mathcal{F} , that is the sub- \mathcal{A} -module *orthogonal* to the \mathcal{A} -module \mathcal{E} . We shall call \mathcal{E}^\perp the **left kernel** sub- \mathcal{A} -module of the pairing $[\mathcal{F}, \mathcal{E}; \mathcal{A}]$. Similarly, we call \mathcal{F}^\perp the **right kernel** sub- \mathcal{A} -module of $[\mathcal{F}, \mathcal{E}; \mathcal{A}]$. Other authors such as Crumeyrolle[[2], p.2] would call \mathcal{E}^\perp (resp. \mathcal{F}^\perp) the **conjugate** of \mathcal{E} (resp. \mathcal{F}) in \mathcal{F} (resp. \mathcal{E}).

Also very important is our attempt, we are concerned with now, of obtaining the *kernel* of an \mathcal{A} -morphism $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ of \mathcal{A} -modules (\mathcal{E} and \mathcal{F} are defined on a topological space X) as the sheafification of some presheaf of $\mathcal{A}(U)$ -modules on X . The kernel of ϕ , denoted here

$$\mathcal{Ker}\phi \equiv (\mathcal{Ker}\phi, \pi|_{\mathcal{Ker}\phi}, X),$$

is a sub- \mathcal{A} -module of $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$, see Mallios[[3], p.108].

Lemma 2.3 *Let \mathcal{E} and \mathcal{F} be \mathcal{A} -modules on a topological space X , $\Gamma(\mathcal{E}) \equiv (\mathcal{E}(U) \equiv \Gamma(U, \mathcal{E}), \pi_V^U)$ the (complete) presheaf of sections of \mathcal{E} , and $\phi \equiv (\phi_U) \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$. Furthermore, for every open subset $U \subseteq X$, let*

$$\ker \phi_U = \{s \in \mathcal{E}(U) : \phi_U(s) = 0 \in \mathcal{F}(U)\}.$$

Then, the diagram

$$\begin{array}{ccc} U & \longrightarrow & \ker \phi_U \\ \uparrow & & \downarrow \lambda_V^U \\ V & \longrightarrow & \ker \phi_V \end{array}$$

where $\lambda_V^U := \pi_V^U|_{\ker \phi_U}$, yields a complete presheaf of $\mathcal{A}(U)$ -modules, denoted

$$\ker \phi := ((\ker \phi)(U) \equiv \ker \phi_U, \lambda_V^U),$$

which is the same as the (complete) presheaf of sections of the kernel $\mathcal{Ker}\phi \equiv (\mathcal{Ker}\phi, \pi|_{\mathcal{Ker}\phi}, X)$ of ϕ .

Proof. That $\ker \phi \equiv (\ker \phi_U, \lambda_V^U)$ is a presheaf of $\mathcal{A}(U)$ -modules is easy to see.

Now, let $s \in (\mathcal{Ker}\phi)(U) \equiv \Gamma(U, \mathcal{Ker}\phi)$, where U is an open subset of X . Evidently, for $x \in U$, $s(x) \in (\mathcal{Ker}\phi)_x = \mathcal{Ker}\phi_x$, see Mallios[[3], (2.11), p.108]. We further obtain that, for all $x \in U$,

$$\phi^*(s)(x) := \phi_x(s(x)) = 0 \in \mathcal{F}_x,$$

see Mallios[[3], Proposition 2.1, p.11]. It thus follows that $\phi^*(s) = 0$, or equivalently $\phi_U(s) := \phi \circ s = \phi^*(s) = 0$. Hence, $s \in \ker \phi_U$, so that $(\mathcal{Ker}\phi)(U) \subseteq \ker \phi_U$.

Conversely, let $s \in \ker \phi_U \subset \mathcal{E}(U) \equiv \Gamma(U, \mathcal{E})$. Then, $\phi_x(s(x)) := \phi_U(s)(x) = 0$ for $x \in U$; consequently $s(x) \in (\mathcal{Ker}\phi)_x$ for $x \in U$. Since $s \in \mathcal{E}(U)$, it follows that $s \in (\mathcal{Ker}\phi)(U)$. Thus, $\ker \phi_U \subseteq (\mathcal{Ker}\phi)(U)$ for every open $U \subseteq X$; hence $\ker \phi_U = (\mathcal{Ker}\phi)(U)$ for every open $U \subseteq X$.

We deduce from the above that $\ker \phi = \Gamma(\mathcal{Ker}\phi)$, and the proof is finished. ■

We are ready now for one important result regarding pairings of \mathcal{A} -modules. Let \mathcal{E} and \mathcal{F} be \mathcal{A} -modules on a topological space X , and

$$\Gamma(\mathcal{E}) \equiv (\Gamma(U, \mathcal{E}), \pi_V^U) \quad \text{and} \quad \Gamma(\mathcal{F}) \equiv (\Gamma(U, \mathcal{F}), \rho_V^U)$$

their corresponding (complete) presheaves of sections, respectively. We also assume that $\Gamma(\mathcal{A}) \equiv (\Gamma(U, \mathcal{A}), \kappa_V^U)$ is the presheaf of sections for the sheaf \mathcal{A} . Suppose that in the pairing $[\mathcal{F}, \mathcal{E}; \mathcal{A}]$ the *left kernel* is 0, i.e. $\mathcal{E}^\perp = 0$. Let U be an open subset of X . For every $r \in \mathcal{F}(U)$, consider the $\mathcal{A}(U)$ -morphism $\phi_r \equiv (\phi_r, V)_{U \supseteq V, \text{ open}} \in \mathcal{E}^*(U)$, given by

$$\phi_r(t) = \rho_V^U(r)t$$

for all $t \in \mathcal{E}(V)$, where V is an open subset of U . One sees that the $\mathcal{A}(U)$ -map

$$\Phi_U : \mathcal{F}(U) \longrightarrow \mathcal{E}^*(U); \quad r \longmapsto \phi_r$$

is an $\mathcal{A}(U)$ -homomorphism. In fact, for all $t \in \mathcal{E}(V)$, where V is open in U , and all $a \in \mathcal{A}(U)$ and $r, s \in \mathcal{F}(U)$, we have

$$\phi_{r+s}(t) = \rho_V^U(r+s)t = (\rho_V^U(r) + \rho_V^U(s))t = \rho_V^U(r)t + \rho_V^U(s)t = \phi_r(t) + \phi_s(t)$$

and

$$\phi_{ar}(t) = \rho_V^U(ar)t = \kappa_V^U(a)\rho_V^U(r)t = \kappa_V^U(a)\phi_r(t) \equiv a\phi_r(t).$$

However,

$$\ker \Phi_U = \{r \in \mathcal{F}(U) : \phi_r(t) = 0 \text{ for all } t \in \mathcal{E}(V), \text{ where } V \text{ is open in } U\}.$$

Since we assumed that $\mathcal{E}^\perp = 0$, i.e. $\mathcal{E}^\perp(U) = \mathcal{E}(U)^\perp = 0$ for all open $U \subseteq X$, it follows that $r = 0$. Hence, for every open $U \subseteq X$, Φ_U is an $\mathcal{A}(U)$ -isomorphism into. If we let U vary over the open subsets of X , the family $\Phi \equiv (\Phi_U)_{X \supseteq U, \text{ open}}$ is a $\Gamma(\mathcal{A})$ -morphism of presheaves $\Gamma(\mathcal{F})$ and $\Gamma(\mathcal{E}^*)$. In fact, first observe that if $\Gamma(\mathcal{E}^*) \equiv (\mathcal{E}^*(U), \pi_V^{*U})$, the restriction maps are defined as follows: For $\alpha \equiv (\alpha_O)_{U \supseteq O, \text{ open}} \in \mathcal{E}^*(U) := \text{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)$,

$$\pi_V^{*U}(\alpha) := (\alpha_O)_{V \supseteq O, \text{ open}} \in \mathcal{E}^*(V).$$

Hence, for $r \in \mathcal{F}(U)$ and $t \in (\mathcal{E}|_V)(W) = \mathcal{E}(W)$, where W is open in V , we have

$$\begin{aligned} \Phi_V(\rho_V^U(r))(t) &= \phi_{\rho_V^U(r)}(t) \\ &= \rho_W^V(\rho_V^U(r))(t) \\ &= \rho_W^U(r)(t) \end{aligned}$$

and

$$\begin{aligned} \pi_V^{*U}(\Phi_U(r))(t) &= \pi_V^{*U}(\phi_r)(t) \\ &= (\phi_{r,O})_{V \supseteq O, \text{ open}}(t) \\ &= \rho_W^U(r)(t); \end{aligned}$$

thus the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\Phi_U} & \mathcal{E}^*(U) \\ \rho_V^U \downarrow & & \downarrow \pi_V^{*U} \\ \mathcal{F}(V) & \xrightarrow{\Phi_V} & \mathcal{E}^*(V) \end{array}$$

commutes, and consequently $\Phi \in \text{Hom}_{\Gamma(\mathcal{A})}(\Gamma(\mathcal{F}), \Gamma(\mathcal{E}^*))$. Through the sheafification functor \mathbf{S} , and on the basis of Mallios[[3], (13.19), p.75], we have that

$$\mathbf{S}(\Phi) \equiv \tilde{\Phi} \in \text{Hom}_{\mathcal{S}h_X}(\mathcal{F}, \mathcal{E}^*)$$

is an \mathcal{A} -isomorphism into.

Similarly, suppose that $\mathcal{F}^\perp = 0$; let Ψ_U be the $\mathcal{A}(U)$ -map, given by

$$\Psi_U : \mathcal{E}(U) \longrightarrow \mathcal{F}^*(U); \quad s \longmapsto \Psi_U(s) \equiv \psi_s$$

where

$$\psi_s(t) = t\pi_V^U(s)$$

for all $t \in \mathcal{F}(V)$ and an open $V \subseteq U$. For $r, s \in \mathcal{E}(U)$, $a \in \mathcal{A}(U)$ and $t \in \mathcal{F}(V)$, where as above V is open in U , we have

$$\psi_{r+s}(t) = t(\pi_V^U(r) + \pi_V^U(s)) = t\pi_V^U(r) + t\pi_V^U(s) = \psi_r(t) + \psi_s(t)$$

and

$$\psi_{ra}(t) = t\pi_V^U(ra) = t\pi_V^U(r)\kappa_V^U(a) = \psi_r(t)\kappa_V^U(a) \equiv \psi_r(t)a;$$

so that Ψ_U is an $\mathcal{A}(U)$ -homomorphism. Now, since $\mathcal{F}^\perp(U) = \mathcal{F}(U)^\perp = 0$ for every open $U \subseteq X$,

$$\ker \Psi_U := \{s \in \mathcal{E}(U) : \psi_s(t) = 0 \text{ for all } t \in \mathcal{F}(V) \text{ and } V \text{ open in } U\} = 0,$$

for any open $U \subseteq X$. Hence, every Ψ_U is an $\mathcal{A}(U)$ -isomorphism into. As in the previous case, the family $\Psi \equiv (\Psi_U)_{X \supseteq U, \text{ open}}$ is a $\Gamma(\mathcal{A})$ -isomorphism into of presheaves $\Gamma(\mathcal{E})$ and $\Gamma(\mathcal{F}^*)$. Consequently,

$$\mathbf{S}(\Psi) \equiv \tilde{\Psi} \in \text{Hom}_{\mathcal{S}h_X}(\mathcal{E}, \mathcal{F}^*)$$

is an \mathcal{A} -isomorphism into.

Let us make a short breach here for the following useful lemmas.

Lemma 2.4 *Let (X, \mathcal{A}) be a \mathbb{C} -algebraized space, and $E \equiv (E(U), \rho_V^U)$ and $F \equiv (F(U), \sigma_V^U)$ be presheaves of $\mathcal{A}(U)$ -modules on X . Then,*

$$\mathbf{S}(E \times F) = \mathbf{S}(E) \times \mathbf{S}(F)$$

within an \mathcal{A} -isomorphism.

Proof. Sheaves $\mathbf{S}(E \times F)$ and $\mathbf{S}(E) \times \mathbf{S}(F)$ clearly have the same stalks at every $x \in X$. Therefore the underlying sets of $\mathbf{S}(E \times F)$ and $\mathbf{S}(E) \times \mathbf{S}(F)$ are the same. It remains only to show that the topology making $\mathbf{S}(E \times F)$ into a sheaf is the same as the topology which defines the sheaf structure on $\mathbf{S}(E) \times \mathbf{S}(F)$. The topology of $\mathbf{S}(E \times F)$ is the topology generated by the basis

$$\{\tilde{s}(V) : s \in (E \times F)(U), \text{ where } U, V \text{ are open in } X \text{ with } V \subseteq U\},$$

see Mallios[[3], Theorem 3.1, p.14]. But $s \in (E \times F)(U) = E(U) \times F(U)$ is if and only if it is of the form

$$s = (s_1, s_2)$$

where $s_1 \in E(U)$ and $s_2 \in F(U)$. It follows that

$$\tilde{s} = (\tilde{s}_1, \tilde{s}_2);$$

consequently

$$\tilde{s}(V) = (\tilde{s}_1(V), \tilde{s}_2(V))$$

for any open subset $V \subseteq U \equiv \text{Dom}(\tilde{s})$. Besides,

$$\{\tilde{s}_1(V) : s_1 \in E(U) \text{ and } V \text{ is open in } U\}$$

and

$$\{\tilde{s}_2(V) : s_1 \in F(U) \text{ and } V \text{ is open in } U\}$$

are bases for the topologies of $\mathbf{S}(E)$ and $\mathbf{S}(F)$, respectively, therefore the topology of $\mathbf{S}(E \times F)$ is equivalent to the topology of $\mathbf{S}(E) \times \mathbf{S}(F)$; thus the proof is finished. ■

Lemma 2.5 *Let $E \equiv (E(U), \rho_V^U)$, $F \equiv (F(U), \sigma_V^U)$ and $A \equiv (A(U), \kappa_V^U)$ be presheaves of $A(U)$ -modules on a topological space X . Suppose that a map $\phi \in \text{Hom}_{\mathcal{PSh}_X}(E \times F, A)$ is given, and $\mathcal{E} := \mathbf{S}(E)$, $\mathcal{F} := \mathbf{S}(F)$, and $\mathcal{A} := \mathbf{S}(A)$, where $\mathbf{S} : \mathcal{PSh}_X \rightarrow \mathcal{Sh}_X$ is the sheafification functor. Then, if ϕ is bilinear, i.e. every $\phi_U : E(U) \times F(U) \rightarrow A(U)$, where U is open in X , is bilinear, the \mathcal{A} -morphism $\mathbf{S}(\phi) \equiv \bar{\phi} \in \text{Hom}_{\mathcal{Sh}_X}(\mathcal{E} \times \mathcal{F}, \mathcal{A})$ is also bilinear, and*

$$\bar{\phi}_U(\tilde{s}, \tilde{t}) = \widetilde{\phi_U(s, t)} \quad (3)$$

for $s \in E(U)$, $t \in F(U)$, $\tilde{s} = \rho_U(s)$, and $\tilde{t} = \sigma_U(t)$, cf. Mallios[[3], (7.22), p.32].

Proof. First, let us make the following comment. Given a presheaf $S \equiv (S(U), \rho_V^U)$ on a topological space X , the sheafification of S hinges for every open subset U of X on the corresponding map $\rho_U : S(U) \rightarrow \Gamma(U, \mathcal{S}) \equiv \mathcal{S}(U)$, which associates with every $s \in S(U)$ the section $\rho_U(s) \equiv \tilde{s} \in \mathcal{S}(U)$. In the special case where S is a presheaf of $A(U)$ -modules, maps ρ_U are

$A(U)$ -homomorphisms, so that $\rho(s+t) \equiv \widetilde{s+t} = \tilde{s} + \tilde{t} \equiv \rho_U(s) + \rho_U(t)$, for all $s, t \in S(U)$. Thus, we have the following.

Given that \mathbf{S} is a functor, the diagram

$$\begin{array}{ccc} E(U) \times F(U) & \xrightarrow{\rho_U \times \sigma_U} & \mathcal{E}(U) \times \mathcal{F}(U) \\ \phi_U \downarrow & & \downarrow \mathbf{S}(\phi) \equiv \bar{\phi} \\ A(U) & \xrightarrow{\kappa_U} & \mathcal{A}(U) \end{array}$$

where $\rho_U : E(U) \rightarrow \mathcal{E}(U)$, $\sigma_U : F(U) \rightarrow \mathcal{F}(U)$, and $\kappa_U : A(U) \rightarrow \mathcal{A}(U)$ are the (canonical) maps defining the respective sheafifications, is commutative, and one has Equation (3). For $s, s' \in E(U)$ and $t \in F(U)$, where U is an open subset of X , it is easy to see that

$$\mathbf{S}(\phi_U)(\widetilde{s+s'}, \tilde{t}) = \mathbf{S}(\phi_U)(\tilde{s}, \tilde{t}) + \mathbf{S}(\phi_U)(\tilde{s'}, \tilde{t}).$$

Likewise one shows linearity in the second component, and that is the end of the proof. ■

Now, let us return to the assumption $\mathcal{E}^\perp = 0$, and let \mathcal{F}_0 be a sub- \mathcal{A} -module of \mathcal{F} . Following Artin[[1], p.19], we find in a natural way a new pairing: more precisely, the *pairing of \mathcal{A} -modules \mathcal{F}_0 and $\mathcal{E}/\mathcal{F}_0^\perp$ into \mathcal{A}* . For this purpose, let U be an open subset of X ; given $t \in \mathcal{F}_0(U)$ and $s \in \mathcal{E}(U)$, we define as product of t and $s + \mathcal{F}_0^\perp(U)$ the element ts of $\mathcal{A}(U)$:

$$t \cdot (s + \mathcal{F}_0^\perp(U)) := ts. \quad (4)$$

That this multiplication is well defined is easy to see. In fact, suppose $s + \mathcal{F}_0^\perp(U) = s_1 + \mathcal{F}_0^\perp(U)$; therefore $s - s_1 \in \mathcal{F}_0^\perp(U)$. But $t \in \mathcal{F}_0(U)$, so $t(s - s_1) = 0$, i.e. $ts = ts_1$. It is also easy to see that the multiplication (4) is bilinear, thus we have the pairing $[\mathcal{F}_0(U), \mathcal{E}(U)/\mathcal{F}_0^\perp(U); \mathcal{A}(U)]$. In turn, if we let U run over the open subsets of X and every multiplication $\mathcal{F}_0(U) \times \mathcal{E}(U)/\mathcal{F}_0^\perp(U) \rightarrow \mathcal{A}(U)$ is given as in Equation (4), we obtain a $\Gamma(\mathcal{A})$ -morphism

$$\begin{array}{ccc} \mathcal{F}_0(U) \times (\mathcal{E}(U)/\mathcal{F}_0^\perp(U)) & \longrightarrow & \mathcal{A}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_0(V) \times (\mathcal{E}(V)/\mathcal{F}_0^\perp(V)) & \longrightarrow & \mathcal{A}(V) \end{array}$$

of $\Gamma(\mathcal{A})$ -presheaves $\Gamma(\mathcal{F}_0) \times \Gamma(\mathcal{E})/\Gamma(\mathcal{F}_0^\perp)$ and $\Gamma(\mathcal{A})$, see Mallios[[3], pp 114-115]. But, by virtue of Lemma 2.4,

$$\mathbf{S}(\Gamma(\mathcal{F}_0) \times \Gamma(\mathcal{E})/\Gamma(\mathcal{F}_0^\perp)) = \mathcal{F}_0 \times (\mathcal{E}/\mathcal{F}_0^\perp)$$

within an \mathcal{A} -isomorphism, it follows through the sheafification functor and from Lemma 2.5 that the \mathcal{A} -morphism

$$\mathcal{F}_0 \times (\mathcal{E}/\mathcal{F}_0^\perp) \longrightarrow \mathcal{A}$$

is bilinear. Hence, we have a pairing $[\mathcal{F}_0, \mathcal{E}/\mathcal{F}_0^\perp; \mathcal{A}]$. Now, what is the *right* kernel of the pairing $[\mathcal{F}_0, \mathcal{E}/\mathcal{F}_0^\perp; \mathcal{A}]$? We shall denote this right kernel by

$$\widehat{\mathcal{F}_0}$$

to differentiate it from the orthogonal \mathcal{F}_0^\perp . First, let us observe the following. Let U be an open subset of X ; the right kernel of the pairing $[\mathcal{F}_0(U), \mathcal{E}(U)/\mathcal{F}_0^\perp(U); \mathcal{A}(U)]$, which we denote by $\widehat{\mathcal{F}_0(U)}$, consists of elements $s + \mathcal{F}_0^\perp(U)$ such that $t \cdot (s + \mathcal{F}_0^\perp(U)) := ts = 0$ for all $t \in \mathcal{F}_0(U)$. This means that $s \in \mathcal{F}_0(U)^\perp = \mathcal{F}_0^\perp(U)$ and, therefore $s + \mathcal{F}_0^\perp(U) = \mathcal{F}_0^\perp(U)$. Thus, $\widehat{\mathcal{F}_0(U)} = 0$.

Now, we will show that $\widehat{\mathcal{F}_0} = 0$. Indeed, let $\tilde{s} \in \widehat{\mathcal{F}_0}(U) \subseteq (\mathcal{E}/\mathcal{F}_0^\perp)(U)$. There exists $s + \mathcal{F}_0^\perp(U)$ such that (see Mallios[[3], (7.9), p.30])

$$\tilde{s}(x) = [s + \mathcal{F}_0^\perp(U)]_x$$

for $x \in U$. Let $t \in \mathcal{F}_0(U)$. We have

$$(t\tilde{s})(x) = t(x)\tilde{s}(x) = t(x)[s + \mathcal{F}_0^\perp(U)]_x = 0;$$

since t is arbitrary, $s \in \mathcal{F}_0^\perp(U)$, so that $\tilde{s} = 0$. Therefore, the right kernel of our pairing is 0, as desired, and $\widehat{\mathcal{F}_0}$ is the sheafification of the presheaf

$$\begin{array}{ccc} U & \longrightarrow & \widehat{\mathcal{F}_0(U)} \\ \uparrow & & \downarrow \\ V & \longrightarrow & \widehat{\mathcal{F}_0(V)} \end{array}$$

Furthermore, for every open $U \subseteq X$, we can construct, using the previously established method, an $\mathcal{A}(U)$ -isomorphism into

$$\mathcal{E}(U)/\mathcal{F}_0^\perp(U) \longrightarrow \mathcal{F}_0^*(U);$$

consequently through the sheafification functor we have an \mathcal{A} -isomorphism into:

$$\mathcal{E}/\mathcal{F}_0^\perp \longrightarrow \mathcal{F}_0^*.$$

If \mathcal{E}_0 is a given sub- \mathcal{A} -module of \mathcal{E} , we can also define a natural pairing $[\mathcal{E}_0^\perp, \mathcal{E}/\mathcal{E}_0; \mathcal{A}]$ by setting

$$t \cdot (s + \mathcal{E}_0(U)) = ts, \quad t \in \mathcal{E}_0^\perp(U) = \mathcal{E}_0(U)^\perp, \quad s + \mathcal{E}_0(U) \in \mathcal{E}(U)/\mathcal{E}_0(U).$$

Likewise, we obtain an \mathcal{A} -isomorphism into:

$$\mathcal{E}_0^\perp \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*.$$

We may formulate our results as follows.

Theorem 2.3 *Let \mathcal{F} and \mathcal{E} be \mathcal{A} -modules on a topological space X paired into a \mathbb{C} -algebra sheaf \mathcal{A} , and assume that $\mathcal{E}^\perp = 0$. Moreover, let \mathcal{F}_0 be a sub- \mathcal{A} -module of \mathcal{F} and \mathcal{E}_0 a sub- \mathcal{A} -module of \mathcal{E} . There exist natural \mathcal{A} -isomorphisms into:*

$$\mathcal{E}/\mathcal{F}_0^\perp \longrightarrow \mathcal{F}_0^*, \quad \text{and} \quad \mathcal{E}_0^\perp \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*.$$

Now, as a special case assume in the pairing $[\mathcal{F}, \mathcal{E}; \mathcal{A}]$ of Theorem 2.3 that both \mathcal{F} and \mathcal{E} are free \mathcal{A} -modules of finite rank m and n respectively. Suppose that $\phi : \mathcal{F} \times \mathcal{E} \longrightarrow \mathcal{A}$ is the bilinear morphism which defines the pairing. Define \mathcal{A} -morphisms $\gamma : \mathcal{E} \longrightarrow \mathcal{F}^* = \mathcal{F}$ and $\delta : \mathcal{F} \longrightarrow \mathcal{E}^* = \mathcal{E}$ (see Mallios[[3], (5.2), p. 298]) such that

$$\gamma_U(s)(t) \equiv \gamma_U^s(t) := \phi_U(t, s)$$

and

$$\delta_U \equiv \delta_U^t(s) := \phi_U(t, s)$$

for all $s \in \mathcal{E}(U)$ and $t \in \mathcal{F}(U)$, where U is an open subset of X . For every open subset U of X , $\ker \gamma_U = \mathcal{F}^\perp(U)$ and $\ker \delta_U = \mathcal{E}^\perp(U)$; therefore

$$\mathcal{Ker} \gamma = \mathcal{F}^\perp, \quad \mathcal{Ker} \delta = \mathcal{E}^\perp$$

within \mathcal{A} -isomorphisms respectively. The \mathcal{A} -morphism ϕ is said to be **non-degenerate** if $\mathcal{E}^\perp = \mathcal{F}^\perp = 0$, and **degenerate** otherwise.

If $\mathcal{E}^\perp = \mathcal{F}^\perp = 0$, γ and δ are injective, or equivalently $\gamma_x : \mathcal{E}_x \longrightarrow \mathcal{F}_x$ and $\delta_x : \mathcal{F}_x \longrightarrow \mathcal{E}_x$ are injective for every $x \in X$, cf. Mallios[[3], Lemma 12.1, p.60]. It follows clearly that $m = n$, i.e. \mathcal{F} is \mathcal{A} -isomorphic to \mathcal{E} .

On the other hand, suppose that \mathcal{E}^\perp and \mathcal{F}^\perp are not all zero. For this case, we will need the following lemma.

Lemma 2.6 *Let $A \equiv (A(U), \kappa_V^U)$ be a presheaf of \mathbb{C} -algebras on a topological space X , $E \equiv (E(U), \rho_V^U)$ and $F \equiv (F(U), \sigma_V^U)$ A -presheaves (i.e. presheaves of $A(U)$ -modules) on X , and finally $\Phi \equiv (\Phi_U)_{X \supseteq U, \text{ open}}$ a bilinear A -morphism $\Phi : F \times E \longrightarrow A$. For every open $U \subseteq X$, let*

$$E(U)^\perp \equiv E^\perp(U) := \{t \in F(U) : \Phi_U(t, s) = 0 \text{ for all } s \in E(U)\}.$$

Assume that $E^\perp \equiv (E^\perp \equiv E(U)^\perp, \lambda_V^U \equiv \sigma_V^U|_{E^\perp(U)})$ is a presheaf on X . Then, if $\mathcal{E} \equiv \mathbf{S}(E)$ and $\mathcal{A} \equiv \mathbf{S}(A)$, one has that

$$\mathcal{E}^\perp = \mathbf{S}(E^\perp)$$

within an \mathcal{A} -isomorphism.

Proof. Let U be an open subset of X ; $\mathbf{S}(E^\perp)(U)$ consists of elements (in fact (local) sections) $\tilde{t} \equiv \rho_U(t)$, where $t \in E^\perp(U)$ and $\rho_U : E^\perp(U) \longrightarrow \Gamma(U, \mathbf{S}(E^\perp)) \equiv \mathbf{S}(E^\perp)(U)$ is the canonical map obtained through the sheafification process, see Mallios[[3], (7.22), p. 32]. Let $\mathcal{F} \equiv \mathbf{S}(F)$, and denote by $\overline{\Phi}$ the bilinear \mathcal{A} -morphism $\overline{\Phi} : \mathcal{F} \times \mathcal{E} \longrightarrow \mathcal{A}$ induced by the presheaf morphism Φ , see Lemma 2.5. But, for all $t \in E^\perp(U)$ and $s \in E(U)$, we have

$$\overline{\Phi}_U(\tilde{s}, \tilde{t}) = \widetilde{\Phi_U(s, t)} = 0,$$

which means that $\tilde{t} \in \mathbf{S}(E)(U)$ if and only if $\tilde{t} \in \mathcal{E}^\perp(U)$; hence $\mathbf{S}(E)(U) = \mathcal{E}^\perp(U)$, and the proof is finished. ■

Getting back to the case where \mathcal{E}^\perp and \mathcal{F}^\perp are not all zero in the pairing $[\mathcal{F}, \mathcal{E}; \mathcal{A}]$, defined by the map $\Phi : \mathcal{F} \times \mathcal{E} \longrightarrow \mathcal{A}$, and where \mathcal{F} and \mathcal{E} are free \mathcal{A} -modules of finite rank on X , we notice that for every open subset U of X , if $t, t_1 \in \mathcal{F}(U)$, and $t - t_1 \in \mathcal{E}^\perp(U)$, then $\Phi_U(t, s) = \Phi_U(t_1, s)$ for all $s \in \mathcal{E}(U)$. Analogously if $s, s_1 \in \mathcal{E}(U)$ and $t, t_1 \in \mathcal{F}(U)$ such that $s - s_1 \in \mathcal{F}^\perp(U)$ and $t - t_1 \in \mathcal{E}^\perp(U)$, we have

$$\Phi_U(t, s) = \Phi_U(t_1, s_1).$$

Setting

$$\bar{s} \equiv [s](\text{mod } \mathcal{F}^\perp(U)), \quad \bar{t} \equiv [t](\text{mod } \mathcal{E}^\perp(U))$$

we obtain the bilinear $\mathcal{A}(U)$ -morphism

$$\bar{\Phi}_U : \mathcal{F}(U)/\mathcal{E}^\perp(U) \times \mathcal{E}(U)/\mathcal{F}^\perp(U) \longrightarrow \mathcal{A}(U),$$

given by

$$\bar{\Phi}_U(\bar{t}, \bar{s}) = \Phi_U(t, s)$$

for all $t \in \mathcal{F}(U)$ and $s \in \mathcal{E}(U)$. Clearly, that $\bar{\Phi}_U(\bar{t}, \bar{s}) = 0$ for all $\bar{s} \in \mathcal{E}(U)/\mathcal{F}^\perp(U)$ is equivalent to saying that $\Phi_U(t, s) = 0$ for all $s \in \mathcal{E}(U)$, so that $t \in \mathcal{E}^\perp(U)$, and $\bar{t} = 0$. Similarly, $\bar{\Phi}_U(\bar{t}, \bar{s}) = 0$ for all $\bar{t} \in \mathcal{F}(U)/\mathcal{E}^\perp(U)$ is equivalent to $\bar{s} = 0$. Hence, $(\mathcal{F}(U)/\mathcal{E}^\perp(U))^\perp = 0$ and $(\mathcal{E}(U)/\mathcal{F}^\perp(U))^\perp = 0$, i.e. $\bar{\Phi}_U$ is non-degenerate. By Lemma 2.6,

$$(\mathcal{F}/\mathcal{E}^\perp)^\perp \equiv \mathbf{S}(\Gamma(\mathcal{F})/\Gamma(\mathcal{E}^\perp))^\perp = \mathbf{S}((\Gamma(\mathcal{F})/\Gamma(\mathcal{E}^\perp))^\perp);$$

since $(\mathcal{F}(U)/\mathcal{E}^\perp(U))^\perp = 0$ for all open subset $U \subseteq X$, it follows that $(\mathcal{F}/\mathcal{E}^\perp)^\perp = 0$. Likewise $(\mathcal{E}/\mathcal{F}^\perp)^\perp = 0$. Thus, we have

Theorem 2.4 *Let (X, \mathcal{A}) be a \mathbb{C} -algebraized space, \mathcal{F} and \mathcal{E} be free \mathcal{A} -modules of finite rank, paired into \mathcal{A} through a bilinear morphism $\Phi : \mathcal{F} \times \mathcal{E} \longrightarrow \mathcal{A}$. The following hold:*

- (i) *If Φ is non-degenerate, then $\mathcal{F} = \mathcal{E}$ within an \mathcal{A} -isomorphism.*

- (ii) If Φ is degenerate, then $\mathcal{F}/\mathcal{E}^\perp$ and $\mathcal{E}/\mathcal{F}^\perp$ are paired into \mathcal{A} through the \mathcal{A} -morphism $\overline{\Phi} : \mathcal{F}/\mathcal{E}^\perp \times \mathcal{E}/\mathcal{F}^\perp \longrightarrow \mathcal{A}$, given by

$$\overline{\Phi}_U(\bar{t}, \bar{s}) = \Phi_U(t, s)$$

for all $t \in \mathcal{F}(U)$, $s \in \mathcal{E}(U)$, and where U is an arbitrary open subset of X . Moreover, $\overline{\Phi}$ is non-degenerate, i.e.

$$(\mathcal{F}/\mathcal{E}^\perp)^\perp = 0, \quad (\mathcal{E}/\mathcal{F}^\perp)^\perp = 0.$$

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